On the number of pairs of positive integers $x_1, x_2 \le H$ such that $x_1 x_2$ is a k-th power

D. I. Tolev*

Abstract

We find an asymptotic formula for the number of pairs of positive integers $x_1, x_2 \le H$ such that the product $x_1 x_2$ is a k-th power.

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1 Notations

Let H be a sufficiently large positive number and $k \ge 2$ be a fixed integer. By the letters $j, l, m, n, u, v, x, y, z$ we denote positive integers. The letter p is reserved for primes and, respectively, $\prod_p$ denotes a product over all primes. By the letters s and w we denote complex numbers and $i = \sqrt{-1}$. By ε we denote an arbitrary small positive number. The constants in the Vinogradov and Landau symbols are absolute or depend on ε and k. As usual, $\zeta(s)$ is the Riemann zeta function. By V_k we denote the set of k-free numbers (i.e. positive integers not divided by a k-th power of a prime) and N_k is the set of k-th powers of natural numbers. We denote by $\mu(n)$ the Möbius function and by $\tau(n)$ the number of positive divisors of n. Further, we define $\eta(n) = \prod_{p|n} p$. We write (u, v) for the greatest common divisor of u and v. We assume that $\min(1, 0^{-1}) = 1$. Finally, by $\Box$ we mark an end of a proof or its absence.

2 Introduction and statement of the result

Let $S_k(H)$ be the number of pairs of positive integers $x_1, x_2 \le H$ such that $x_1 x_2 \in N_k$. In the the present paper we establish an asymptotic formula for $S_k(H)$. This problem is related to a result of Heath-Brown and Moroz [2]. They considered in 1999 the diophantine

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equation $x_1x_2x_3 = x_0^3$ and found an asymptotic formula for the number of primitive solutions such that $1 \le x_1, x_2, x_3 \le H$.

First we note that it is easy to find an asymptotic formula for the quantity

$$S_k^*(H) = \#\{x_1, x_2 \ : \ x_1, x_2 \le H, \quad (x_1, x_2) = 1, \quad x_1x_2 \in N_k\}.$$

Indeed, if $(x_1, x_2) = 1$ then $x_1x_2 \in N_k$ exactly when $x_1 \in N_k$ and $x_2 \in N_k$. Hence

$$S_k^*(H) = \#\{x_1, x_2 \ : \ x_1, x_2 \le H, \quad (x_1, x_2) = 1, \quad x_1 \in N_k, \quad x_2 \in N_k\} = \sum_{\substack{z_1, z_2 \le H^{1/k} \\ (z_1, z_2) = 1}} 1$$

and using the well-known property of the Möbius function we get

$$S_k^*(H) = \sum_{z_1, z_2 \le H^{1/k}} \ \sum_{d | (z_1, z_2)} \mu(d) = \sum_{d \le H^{1/k}} \mu(d) \left(\frac{H^{1/k}}{d} + O(1) \right)^2 .$$

Therefore

$$S_k^*(H) = H^{2/k} \sum_{d \le H^{1/k}} \frac{\mu(d)}{d^2} + O\left(H^{1/k} \log H\right) = \zeta(2)^{-1} H^{2/k} + O\left(H^{1/k} \log H\right). \qquad (1)$$

We remark also that it is easy to evaluate $S_2(H)$. Indeed, we have

$$S_2(H) = \sum_{d \le H} \ \sum_{\substack{x_1, x_2 \le H \\ (x_1, x_2) = d \\ x_1x_2 \in N_2}} 1 = \sum_{d \le H} \ \sum_{\substack{y_1, y_2 \le H/d \\ (y_1, y_2) = 1 \\ y_1y_2d^2 \in N_2}} 1 = \sum_{d \le H} S_2^*(H/d).$$

Now we apply (1) and after certain calculations, which we leave to the reader, we find

$$S_2(H) = \zeta(2)^{-1} H \log H + O\left(H\right).$$

However it is not clear how to apply (1) in order to evaluate $S_k(H)$ for $k \ge 3$.

Another quantity related to $S_k(H)$ is

$$T_k(H) = \#\{x_1, x_2 \ : \ x_1x_2 \le H^2, \quad x_1x_2 \in N_k\} = \sum_{n \le H^{2/k}} \tau\left(n^k\right).$$

Using well-known analytic methods, based on Perron's formula and the simplest properties of $\zeta(s)$, we are able to prove the asymptotic formula

$$T_k(H) \sim \gamma_k \, H^{2/k} \, (\log H)^k,$$

where $\gamma_k > 0$ depends only on k. In the present paper we show that using the same analytic tools, as well as an idea of Heath-Brown and Moroz [2], we may find an asymptotic formula for $S_k(H)$ for any $k \ge 2$. Our result is the following theorem.

Theorem. *For any integer $k \geq 2$ we have*

$$S_k(H) = c_k H^{2/k}(\log H)^{k-1} + O\left(H^{2/k}(\log H)^{k-2}\right), \tag{2}$$

where

$$c_k = \frac{\mathcal{P}_k}{((k-1)!)^2}\left(1 + \frac{1}{k^{k-2}}\sum_{k/2<m\leq k-1}\frac{(-1)^{k-m}\,(2m-k)^{k-1}\binom{k-1}{m}}{k-m}\right), \tag{3}$$

$$\mathcal{P}_k = \prod_p \left(1-\frac{1}{p}\right)^{k-1}\left(1+\frac{k-1}{p}\right). \tag{4}$$

3 Some lemmas

We need the following elementary

Lemma 1.

(i) *Every positive integer x can be represented uniquely in the form $x = yz$, where $y \in V_k$ and $z \in N_k$.*

(ii) *Every integer $y \in V_k$ can be represented uniquely in the form $y = u_1 u_2^2 u_3^3 \dots u_{k-1}^{k-1}$, where $u_j \in V_2$ for $1 \leq j \leq k-1$ and $(u_i, u_j) = 1$ for $1 \leq i, j \leq k-1$, $i \neq j$.*

(iii) *If $y_1, y_2 \in V_k$ and $y_1 y_2 \in N_k$ then $\eta(y_1) = \eta(y_2) = (y_1 y_2)^{1/k}$.*

Proof: The proofs of (i) and (ii) can by obtained easily from the fundamental theorem of arithmetics and we leave this to the reader. Let us prove (iii). By our assumption, any prime in the factorization of $y_1 y_2$ occurs with exponent at most $2k-2$, hence with exponent exactly k. As the exponent of each prime in y_1 and y_2 is $\leq k-1$, the integers y_1 and y_2 have the same prime factors. □

The next lemma is a version of the Perron formula. Denote

$$E(\gamma) = \begin{cases} 1 & \text{if} \quad \gamma \geq 1, \\ 0 & \text{if} \quad 0 < \gamma < 1. \end{cases} \tag{5}$$

We have

Lemma 2. *If $\gamma > 0$, $0 < c < c_0$ and $T > 1$ then*

$$E(\gamma) = \frac{1}{2\pi i}\int_{c-iT}^{c+iT}\frac{\gamma^s}{s}\,ds + O\left(\gamma^c \min\left(1, T^{-1}|\log\gamma|^{-1}\right)\right).$$

The constant in the Landau symbol depends only on c_0.

Proof: This is a slightly simplified version of a lemma from [1], Section 17. $\Box$.

Some of the basic properties of Riemann's zeta function are presented in the next lemma.

Lemma 3.

(i) $\zeta(s)$ is meromorphic in the complex plane and has a pole only at $s = 1$. It is simple and with a residue equal to 1.

(ii) If $Re(s) > 1$ then $\zeta(s) = \prod_p (1 - p^{-s})^{-1}$.

(iii) If $Re(s) \geq \sigma > 1$ then $\zeta(s) \ll (\sigma - 1)^{-1} + 1$.

(iv) If $1/2 \leq \sigma_0 \leq 1$, $\sigma \geq \sigma_0$ and $|t| \geq 2$ then $\zeta(\sigma + it) \ll |t|^{\frac{1-\sigma_0}{2}+\varepsilon}$.

(v) There exist $\lambda_0 > 0$ such that if $X \geq 2$, $|t| \leq X$ and $\sigma \geq 1 - \frac{\lambda_0}{\log X}$ then $\zeta(\sigma + it) \neq 0$.

Proof: See [3], Chapters 1 – 3 and 5. $\Box$

4 Proof of the theorem

1. We already considered the case $k = 2$, so we may assume that $k \geq 3$.

Working as in [2] we apply Lemma 1 (i) and find that $S_k(H)$ is equal to the number of quadruples y_1, y_2, z_1, z_2 such that

$$y_1, y_2 \in V_k; \quad z_1, z_2 \in N_k; \quad y_1 z_1 \leq H; \quad y_2 z_2 \leq H; \quad y_1 z_1 y_2 z_2 \in N_k.$$

Obviously the last of the above conditions is equivalent to $y_1 y_2 \in N_k$ because z_1 and z_2 are k-th powers. Hence

$$S_k(H) = \sum_{\substack{y_1, y_2 \leq H \\ y_1, y_2 \in V_k \\ y_1 y_2 \in N_k}} \sum_{\substack{m_j \leq (H/y_j)^{1/k} \\ j=1,2}} 1 = \sum_{\substack{y_1, y_2 \leq H \\ y_1, y_2 \in V_k \\ y_1 y_2 \in N_k}} \Big((H/y_1)^{1/k} + O(1) \Big) \Big((H/y_2)^{1/k} + O(1) \Big) .$$

Expanding brackets we get

$$S_k(H) = H^{2/k} U_k(H) + O\left(H^{1/k} W_k(H) \right), \tag{6}$$

where

$$U_k(H) = \sum_{\substack{y_1, y_2 \leq H \\ y_1, y_2 \in V_k \\ y_1 y_2 \in N_k}} (y_1 y_2)^{-1/k}, \qquad W_k(H) = \sum_{\substack{y_1, y_2 \leq H \\ y_1, y_2 \in V_k \\ y_1 y_2 \in N_k}} y_1^{-1/k}.$$

Using Lemma 1 (iii) we see that for a given y_1 the integer y_2 is determined uniquely. Therefore we have

$$U_k(H) = \sum_{\substack{y\le H\\ y\in V_k\\ \eta(y)^k\le Hy}} \eta(y)^{-1}, \qquad W_k(H) = \sum_{\substack{y\le H\\ y\in V_k\\ \eta(y)^k\le Hy}} y^{1/k}\eta(y)^{-1}. \tag{7}$$

To prove the theorem we have to find an asymptotic formula for $U_k(H)$ and to estimate $W_k(H)$.

2. Consider first $W_k(H)$. Applying Lemma 1 (ii) we get

$$W_k(H) \le \sum_{u_1u_2^2\dots u_{k-1}^{k-1}\le H} \frac{\left(u_1u_2^2\dots u_{k-1}^{k-1}\right)^{1/k}}{u_1u_2\dots u_{k-1}}$$

$$= \sum_{u_1u_2^2\dots u_{k-2}^{k-2}\le H} u_1^{-1+1/k}u_2^{-1+2/k}\dots u_{k-2}^{-1+(k-2)/k} \sum_{u_{k-1}\le\left(\frac{H}{u_1u_2^2\dots u_{k-2}^{k-2}}\right)^{1/(k-1)}} u_{k-1}^{-1/k}.$$

The inner sum is $\ll H^{1/k}\left(u_1u_2^2\dots u_{k-2}^{k-2}\right)^{-1/k}$, hence

$$W_k(H) \ll H^{1/k} \sum_{u_1u_2^2\dots u_{k-2}^{k-2}\le H} (u_1u_2\dots u_{k-2})^{-1} \ll H^{1/k}(\log H)^{k-2}. \tag{8}$$

It remains to show that

$$U_k(H) = c_k(\log H)^{k-1} + O\left((\log H)^{k-2}\right). \tag{9}$$

Formula (2) is a consequence of (6), (8) and (9).

3. Using (5) and (7) we write $U_k(H)$ in the form

$$U_k(H) = \sum_{\substack{y\le H\\ y\in V_k}} \eta(y)^{-1}\, E\left(H\, y\, \eta(y)^{-k}\right).$$

We put

$$c = (\log H)^{-1}, \qquad T = (\log H)^{100\,k^3} \tag{10}$$

and applying Lemma 2 we find that

$$U_k(H) = U^{(1)} + O\left(\Delta\right), \tag{11}$$

where

$$U^{(1)} = \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \frac{H^s}{s}\, \Phi(s)\, ds, \qquad \Phi(s) = \sum_{\substack{y \le H \\ y \in V_k}} y^s\, \eta(y)^{-ks-1} \tag{12}$$

and

$$\Delta = \sum_{\substack{y \le H \\ y \in V_k}} \eta(y)^{-1} \min\Big(1, T^{-1} \left|\log\left(H\, y\, \eta(y)^{-k}\right)\right|^{-1}\Big).$$

4. Consider first the sum Δ. We put

$$\varkappa = T^{-1/2} \tag{13}$$

and write

$$\Delta = \Delta_1 + \Delta_2, \tag{14}$$

where in Δ_1 the summation is taken over y satisfying $\left|\log\left(H\, y\, \eta(y)^{-k}\right)\right| \ge \varkappa$ and in Δ_2 over the other y. To estimate Δ_1 we apply Lemma 1 (iii), (10) and (13) to find

$$\Delta_1 \ll T^{-1/2} \sum_{\substack{y \le H \\ y \in V_k}} \eta(y)^{-1} \ll T^{-1/2} \sum_{u_1 u_2^2 \ldots u_{k-1}^{k-1} \le H} (u_1 u_2 \ldots u_{k-1})^{-1} \ll \frac{(\log H)^{k-1}}{T^{1/2}} \ll 1. \tag{15}$$

Consider Δ_2. Using its definition and Lemma 1 (iii) we find

$$\begin{aligned} \Delta_2 &\ll \sum_{\substack{u_1, u_2, \ldots, u_{k-1}\, : \\ \left|\log\left(H/(u_1^{k-1} u_2^{k-2} \ldots u_{k-2}^2 u_{k-1})\right)\right| < \varkappa}} (u_1 u_2 \ldots u_{k-1})^{-1} \\ &\ll \sum_{He^{-\varkappa} < u_1^{k-1} u_2^{k-2} \ldots u_{k-2}^2 u_{k-1} < He^{\varkappa}} (u_1 u_2 \ldots u_{k-1})^{-1} \\ &\ll \sum_{u_1^{k-1} u_2^{k-2} \ldots u_{k-2}^2 < 2H} (u_1 u_2 \ldots u_{k-2})^{-1} \sum_{\frac{He^{-\varkappa}}{u_1^{k-1} u_2^{k-2} \ldots u_{k-2}^2} < u_{k-1} < \frac{He^{\varkappa}}{u_1^{k-1} u_2^{k-2} \ldots u_{k-2}^2}} u_{k-1}^{-1}. \end{aligned}$$

To estimate the inner sum we apply the obvious inequality

$$\sum_{a < n \le b} n^{-1} \le a^{-1} + \log(b/a) \qquad (0 < a < b) \tag{16}$$

and find that

$$\Delta_2 \ll \sum_{u_1^{k-1} u_2^{k-2} \ldots u_{k-2}^2 < 2H} \frac{H^{-1} u_1^{k-1} u_2^{k-2} \ldots u_{k-2}^2 + \varkappa}{u_1 u_2 \ldots u_{k-2}} \ll H^{-1}\, \Delta_3 + \varkappa\, (\log H)^{k-2}, \tag{17}$$

where

$$\Delta_3 = \sum_{u_1^{k-1}u_2^{k-2}\ldots u_{k-2}^2<2H} u_1^{k-2}u_2^{k-3}\ldots u_{k-2}. \tag{18}$$

If $k>3$ then

$$\Delta_3 \ll \sum_{u_1^{k-1}u_2^{k-2}\ldots u_{k-3}^3<2H} u_1^{k-2}u_2^{k-3}\ldots u_{k-3}^2 \sum_{u_{k-2}<\left(2H/(u_1^{k-1}u_2^{k-2}\ldots u_{k-3}^3)\right)^{1/2}} u_{k-2}$$

$$\ll H \sum_{u_1^{k-1}u_2^{k-2}\ldots u_{k-3}^3<2H} (u_1u_2\ldots u_{k-3})^{-1} \ll H(\log H)^{k-3}. \tag{19}$$

The last estimate for Δ_3 is obviously true also for $k=3$. From (10), (13) – (15), (17) and (19) we get

$$\Delta \ll (\log H)^{k-3}. \tag{20}$$

5. Consider the expression $\Phi(s)$ defined by (12). Let c and T be specified by (10) and

$$T_1 = 2kT. \tag{21}$$

We apply Lemma 2 again and that if $Re(s)=c$ then

$$\Phi(s) = \frac{1}{2\pi i}\int_{c-iT_1}^{c+iT_1} \frac{H^w}{w}\mathcal{M}(s,w)\,dw + O\left(\Delta^*\right), \tag{22}$$

where

$$\mathcal{M}(s,w) = \sum_{\substack{y=1\\ y\in V_k}}^{\infty} y^{s-w}\,\eta(y)^{-ks-1}, \tag{23}$$

$$\Delta^* = \sum_{\substack{y=1\\ y\in V_k}}^{\infty} \eta(y)^{-kc-1}\,\min\left(1, T_1^{-1}\left|\log(H/y)\right|^{-1}\right). \tag{24}$$

To justify (22) we note that from Euler's identity, (10) and Lemma 3 *(ii)*, *(iii)* it follows

$$\sum_{\substack{y=1\\ y\in V_k}}^{\infty} \eta(y)^{-kc-1} = \prod_p\left(1+\frac{k-1}{p^{kc+1}}\right) \ll \zeta^{k-1}(kc+1) \ll c^{-k+1} \ll (\log H)^{k-1}. \tag{25}$$

Hence the series $\mathcal{M}(s,w)$ is absolutely and uniformly convergent in $Re(s) = Re(w) = c$ because under this assumption we have

$$\mathcal{M}(s,w) \ll \sum_{\substack{y=1\\ y\in V_k}}^{\infty} \eta(y)^{-kc-1}.$$

This completes the verification of (22).

6. Consider the expression Δ^* defined by (24). We write it in the form

$$\Delta^* = \Delta_1^* + \Delta_2^*, \tag{26}$$

where the summation in Δ_1^* is taken over y such that $|\log(H/y)| \geq \varkappa$ and in Δ_2^* over the other y. Using (10), (13), (21) and (25) we find

$$\Delta_1^* \ll T^{-1/2} \sum_{\substack{y=1\\ y\in V_k}}^{\infty} \eta(y)^{-kc-1} \ll (\log H)^{k-1-50k^3} \ll 1. \tag{27}$$

To estimate Δ_2^* we apply Lemma 1 (iii) and (10), (13), (16) to get

$$\begin{aligned}
\Delta_2^* &\ll \sum_{\substack{He^{-\varkappa}<y<He^{\varkappa}\\ y\in V_k}} \eta(y)^{-1} \ll \sum_{He^{-\varkappa}<u_1u_2^2\ldots u_{k-1}^{k-1}<He^{\varkappa}} (u_1u_2\ldots u_{k-1})^{-1}\\
&\ll \sum_{u_2^2u_3^3\ldots u_{k-1}^{k-1}<2H} (u_2u_3\ldots u_{k-1})^{-1} \sum_{\frac{He^{-\varkappa}}{u_2^2u_3^3\ldots u_{k-1}^{k-1}}<u_1<\frac{He^{\varkappa}}{u_2^2u_3^3\ldots u_{k-1}^{k-1}}} u_1^{-1}\\
&\ll \sum_{u_2^2u_3^3\ldots u_{k-1}^{k-1}<2H} \frac{H^{-1}u_2^2u_3^3\ldots u_{k-1}^{k-1}+\varkappa}{u_2u_3\ldots u_{k-1}}\\
&\ll H^{-1}\Delta_3 + 1,
\end{aligned} \tag{28}$$

where Δ_3 is given by (18). Applying (19), (26) – (28) we find

$$\Delta^* \ll (\log H)^{k-3}. \tag{29}$$

We substitute in formula (12) the expression for $\Phi(s)$ given by (22) and find a new form of $U^{(1)}$. Using (10) and (29) we see that the contribution to $U^{(1)}$ coming from Δ^* is

$$\ll (\log H)^{k-3} \int_{-T}^{T} \frac{dt}{\sqrt{c^2+t^2}} \ll (\log H)^{k-2}.$$

Therefore, taking also into account (11) and (20), we find

$$U_k(H) = \frac{1}{(2\pi i)^2} \int_{c-iT}^{c+iT} \frac{H^s}{s} \int_{c-iT_1}^{c+iT_1} \frac{H^w}{w} \mathcal{M}(s,w)\, dw\, ds + O\left((\log H)^{k-2} \right). \qquad (30)$$

7. For a fixed s satisfying $Re(s) = c$ the infinite series $\mathcal{M}(s,w)$, defined by (23), is absolutely and uniformly convergent for $Re(w) \ge c$ and represents a holomorphic function in $Re(w) > c$. Applying Euler's identity we find

$$\mathcal{M}(s,w) = \prod_p \left(1 + p^{-ks-1} \left(p^{s-w} + p^{2(s-w)} + \cdots + p^{(k-1)(s-w)}\right)\right)$$

$$= \prod_p \left(1 + \sum_{j=1}^{k-1} p^{-(k-j)s-jw-1}\right).$$

Using Lemma 3 (ii) we conclude that for $Re(s) = c$, $Re(w) \ge c$ we have

$$\mathcal{M}(s,w) = \mathcal{K}(s,w) \prod_{j=1}^{k-1} \zeta\left((k-j)s + jw + 1\right), \qquad (31)$$

where

$$\mathcal{K}(s,w) = \prod_p \left\{ \left(1 + \sum_{j=1}^{k-1} p^{-(k-j)s-jw-1}\right) \prod_{j=1}^{k-1} \left(1 - p^{-(k-j)s-jw-1}\right) \right\}.$$

It is clear that there exists $\delta = \delta(k) \in (0, 1/100)$ such that in the region

$$Re(s) > -\delta, \qquad Re(w) > -\delta \qquad (32)$$

the function $\mathcal{K}(s,w)$ is holomorphic with respect to s as well as with respect to w and satisfies

$$0 < |\mathcal{K}(s,w)| \ll 1. \qquad (33)$$

We have also

$$\mathcal{K}(0,0) = \mathcal{P}_k, \qquad (34)$$

where $\mathcal{P}_k$ is given by (4).

Suppose that we have a fixed $s = c + it$ with $-T \le t \le T$. From (31), (33) and Lemma 3 (i) we conclude that the function $H^w w^{-1} \mathcal{M}(s,w)$ has a meromorphic continuation to $Re(w) > -\delta$ and that poles may occur only at the points

$$w = 0, \qquad w = \left(1 - \frac{k}{m}\right) s, \qquad 1 \le m \le k-1. \qquad (35)$$

All these points are actually simple poles. Indeed, for $w = 0$ this follows immediately from (33) and Lemma 3 (i), (v). In the case $1 \le m \le k-1$ the point $w = (1 - k/m)s$ is a simple pole of $\zeta((k-m)s + mw + 1)$ and, due to Lemma 3 (v) and (10), it cannot be a pole or zero of $\zeta((k-j)s + jw + 1)$ for $1 \le j \le k-1$, $j \ne m$.

For $1 \le m \le k-1$ we denote by $\mathcal{R}_m(s)$ the residue of $H^w w^{-1} \mathcal{M}(s,w)$ at $w = (1-k/m)s$ and let $\mathcal{R}_0(s)$ be the residue at $w = 0$. A straightforward calculation, based on the above arguments, (33) and Lemma 3 (i), leads to

$$\mathcal{R}_0(s) = \mathcal{K}(s,0) \prod_{j=1}^{k-1} \zeta(js+1), \tag{36}$$

$$\mathcal{R}_m(s) = \frac{H^{\left(1-\frac{k}{m}\right)s}}{(m-k)s} \mathcal{K}\left(s, \left(1 - \frac{k}{m}\right) s\right) \prod_{\substack{j=1 \\ j \ne m}}^{k-1} \zeta\left(k\left(1 - \frac{j}{m}\right) s + 1\right), \quad 1 \le m \le k-1. \tag{37}$$

8. Let us define

$$\theta = \frac{\delta}{2k^3}. \tag{38}$$

Due (10), (21) and since $s = c + it$, where $-T \le t \le T$, we see that all points (35) are inside the rectangle with vertices $c - iT_1$, $-\theta - iT_1$, $-\theta + iT_1$, $c + iT_1$. Applying the residue theorem we find that

$$\int_{c-iT_1}^{c+iT_1} \frac{H^w}{w} \mathcal{M}(s,w)\, dw = 2\pi i \sum_{m=0}^{k-1} \mathcal{R}_m(s) + I_1 + I_2 + I_3,$$

where

$$I_1 = \int_{c-iT_1}^{-\theta-iT_1} \frac{H^w}{w} \mathcal{M}(s,w)\, dw, \qquad I_2 = \int_{-\theta-iT_1}^{-\theta+iT_1} \frac{H^w}{w} \mathcal{M}(s,w)\, dw,$$

$$I_3 = \int_{-\theta+iT_1}^{c+iT_1} \frac{H^w}{w} \mathcal{M}(s,w)\, dw.$$

From the above formula and (30) we get

$$U_k(H) = \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \frac{H^s}{s} \sum_{m=0}^{k-1} \mathcal{R}_m(s)\, ds \; + J_1 + J_2 + J_3 + \; O\left((\log H)^{k-2}\right). \tag{39}$$

Here J_μ are the contributions coming from I_μ, $\mu = 1, 2, 3$ and we will see that we may neglect them.

To estimate J_μ we will first show that if $s = c + it$, where $|t| \le T$, and if w belongs to some of the sets of integration of I_1, I_2 or I_3 then

$$\mathcal{M}(s, w) \ll T^{k^2\theta}. \tag{40}$$

Having in mind (31) and (33), we see that in order to verify this it is enough to establish that for s and w satisfying the above conditions we have

$$\zeta(\lambda) \ll T^{k\theta}, \qquad \text{where} \qquad \lambda = (k-j)s + jw + 1, \qquad 1 \le j \le k-1. \tag{41}$$

If $w = \beta + iT_1$ (or $w = \beta - iT_1$), where $-\theta \le \beta \le c$, then from (10), (21), (38) it follows that for the number λ, given by (41), we have $Re(\lambda) \ge 1 - k\theta$ and $T \ll |Im(\lambda)| \ll T$. Hence the estimate (41) is a consequence of Lemma 3 (iv). Suppose now that $w = -\theta + it_1$, where $|t_1| \le T_1$. From (10), (21), (38) we get $Re(\lambda) \ge 1 - k\theta$ and $|Im(\lambda)| \ll T$. If $|Im(\lambda)| \ge 2$ then the estimate (41) follows again from Lemma 3 (iv). In the case $|Im(\lambda)| < 2$ we use also the inequality $Re(\lambda) \le 1 - \theta/2$ to conclude that $\zeta(\lambda) \ll 1$, so the estimate (41) is true again.

From the definitions of J_μ and (10), (21), (38), (40) we find

$$J_1, J_3 \ll \int_{-T}^{T} \frac{1}{\sqrt{c^2 + t^2}} \int_{-\theta}^{c} \frac{T^{k^2\theta}}{\sqrt{\beta^2 + T_1^2}} \, d\beta \, dt \ll c^{-1} + \log T \ll \log H$$

and

$$J_2 \ll \int_{-T}^{T} \frac{1}{\sqrt{c^2 + t^2}} \int_{-T_1}^{T_1} \frac{H^{-\theta}\, T^{k^2\theta}}{\sqrt{\theta^2 + t_1^2}} \, dt_1 \, dt \ll H^{-\theta} \left(c^{-1} + \log T\right) T^{k^2\theta} \log T \ll 1.$$

This means that the terms J_μ in formula (39) can be omitted indeed. Then using (36), (37) we get

$$U_k(H) = \frac{1}{2\pi i} \left(\mathfrak{N}_0 + \sum_{m=1}^{k-1} \frac{1}{m-k} \, \mathfrak{N}_m \right) + O\left((\log H)^{k-2} \right), \tag{42}$$

where

$$\mathfrak{N}_m = \int_{c-iT}^{c+iT} \Xi_m(s) \, ds \tag{43}$$

and

$$\Xi_0(s) = s^{-1}\, H^s\, \mathcal{K}(s,0) \prod_{j=1}^{k-1} \zeta(js+1), \tag{44}$$

$$\Xi_m(s) = s^{-2}\, H^{\left(2-\frac{k}{m}\right)s}\, \mathcal{K}\left(s, \left(1-\frac{k}{m}\right)s\right) \prod_{\substack{j=1\\ j\neq m}}^{k-1} \zeta\left(k\left(1-\frac{j}{m}\right)s+1\right), \quad 1\le m\le k-1. \tag{45}$$

9. Consider first $\mathfrak{N}_m$ for $1 \le m \le k/2$. Since $\Xi_m(s)$ is a holomorphic function in the rectangle with vertices $c-iT$, $\theta-iT$, $\theta+iT$, $c+iT$ we have

$$\mathfrak{N}_m = \int_{c-iT}^{\theta-iT} \Xi_m(s)\,ds + \int_{\theta-iT}^{\theta+iT} \Xi_m(s)\,ds + \int_{\theta+iT}^{c+iT} \Xi_m(s)\,ds = \mathfrak{N}_m^{(1)} + \mathfrak{N}_m^{(2)} + \mathfrak{N}_m^{(3)}, \tag{46}$$

say. If s belongs to the sets of integration of $\mathfrak{N}_m^{(1)}$ or $\mathfrak{N}_m^{(3)}$ and if $1 \le j \le k-1$, $j \ne m$ then from Lemma 3 (iv) it follows that $\zeta(k(1-j/m)s+1) \ll T^{k^2\theta}$. Hence, using (33), (38) and our assumption $1 \le m \le k/2$, we find

$$\mathfrak{N}_m^{(1)},\, \mathfrak{N}_m^{(3)} \ll \int_c^\theta \frac{H^{\left(2-\frac{k}{m}\right)\beta}}{\beta^2+T^2}\, T^{k^3\theta}\, d\beta \ll T^{k^3\theta-2} \ll 1. \tag{47}$$

Suppose now that s belongs to the set of integratation of $\mathfrak{N}_m^{(2)}$ (that is $s=\theta+it$, $|t|\le T$) and consider the number $\tilde\lambda = k(1-j/m)s+1$. It is easy to see that for each j that occurs in (45) we have $Re(\tilde\lambda) \ge 1-k^2\theta$, $|Re(\tilde\lambda)-1| \ge \theta$ and $|Im(\tilde\lambda)| \le k^2|t|$. Hence an application of Lemma 3 (iv) gives $\zeta(\tilde\lambda) \ll (1+|t|)^{k^2\theta}$. Therefore

$$\mathfrak{N}_m^{(2)} \ll \int_{-T}^{T} \frac{H^{\left(2-\frac{k}{m}\right)\theta}}{\theta^2+t^2}\, (1+|t|)^{k^3\theta}\, dt \ll 1. \tag{48}$$

From (46) – (48) we get $\mathfrak{N}_m \ll 1$ for $1 \le m \le k/2$ and using (42) we find

$$U_k(H) = \frac{1}{2\pi i}\left(\mathfrak{N}_0 + \sum_{k/2<m\le k-1} \frac{1}{m-k}\,\mathfrak{N}_m\right) + O\left((\log H)^{k-2}\right). \tag{49}$$

10. Consider now $\mathfrak{N}_m$ for $k/2 < m \le k-1$. The function $\Xi_m(s)$ has a pole only at $s=0$ and it is not difficult to compute that the corresponding residue is equal to

$$\mathcal{L}_m\,(\log H)^{k-1} + O\left((\log H)^{k-2}\right),$$

where

$$\mathcal{L}_m = \frac{(2m-k)^{k-1}\,(-1)^{k-m-1}\binom{k-1}{m}\,\mathcal{P}_k}{((k-1)!)^2\,k^{k-2}}. \tag{50}$$

We leave the standard verification to the reader. From (43) and the residue theorem we get

$$\mathfrak{N}_m = 2\pi i\,\mathcal{L}_m\,(\log H)^{k-1} + \mathfrak{N}'_m + \mathfrak{N}''_m + \mathfrak{N}'''_m + O\left((\log H)^{k-2}\right), \tag{51}$$

where

$$\mathfrak{N}'_m = \int_{c-iT}^{-\theta-iT} \Xi_m(s)\,ds, \qquad \mathfrak{N}''_m = \int_{-\theta-iT}^{-\theta+iT} \Xi_m(s)\,ds, \qquad \mathfrak{N}'''_m = \int_{-\theta+iT}^{c+iT} \Xi_m(s)\,ds.$$

Using Lemma 3 (iv) we find that if s belongs to the set of integration of some of the above integrals then the product of the values of the zeta-function in the definition (45) is $\ll T^{k^3\theta}$. Hence from (10), (33), (38) and our assumption $k/2 < m \le k-1$ it follows that

$$\mathfrak{N}'_m,\, \mathfrak{N}'''_m \ll \int_{-\theta}^{c} \frac{T^{k^3\theta}}{\beta^2+T^2}\,d\beta \ll 1 \tag{52}$$

and

$$\mathfrak{N}''_m \ll \int_{-T}^{T} \frac{H^{-\left(2-\frac{k}{m}\right)\theta}}{\theta^2+t^2} T^{k^3\theta}\,dt \ll H^{-\left(2-\frac{k}{m}\right)\theta}\,T^{k^3\theta} \ll 1. \tag{53}$$

From (51) – (53) we find

$$\mathfrak{N}_m = 2\pi i\,\mathcal{L}_m\,(\log H)^{k-1} + O\left((\log H)^{k-2}\right) \qquad \text{for} \qquad k/2 < m \le k-1. \tag{54}$$

11. It remains to consider $\mathfrak{N}_0$. It is not difficult to see that the function $\Xi_0(s)$ specified by (44) has a pole only at $s=0$ with a residue equal to

$$\mathcal{L}_0(\log H)^{k-1} + O\left((\log H)^{k-2}\right),$$

where

$$\mathcal{L}_0 = \frac{\mathcal{P}_k}{((k-1)!)^2}. \tag{55}$$

From (43) and the residue theorem we find

$$\mathfrak{N}_0 = 2\pi i\,\mathcal{L}_0\,(\log H)^{k-1} + \mathfrak{N}'_0 + \mathfrak{N}''_0 + \mathfrak{N}'''_0 + O\left((\log H)^{k-2}\right),$$

where

$$\mathfrak{N}'_0 = \int_{c-iT}^{-\theta-iT} \Xi_0(s)\,ds, \qquad \mathfrak{N}''_0 = \int_{-\theta-iT}^{-\theta+iT} \Xi_0(s)\,ds, \qquad \mathfrak{N}'''_0 = \int_{-\theta+iT}^{c+iT} \Xi_0(s)\,ds.$$

Arguing as above we conclude that $\mathfrak{N}_0'$, $\mathfrak{N}_0''$, $\mathfrak{N}_0''' \ll 1$ (we leave the verification to the reader). Hence

$$\mathfrak{N}_0 = 2\pi i \, \mathcal{L}_0 \, (\log H)^{k-1} + \big((\log H)^{k-2}\big) \, . \tag{56}$$

From (3), (34), (49), (50), (54) – (56) we obtain (9) and the proof of the Theorem is complete. □

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Faculty of Mathematics and Informatics
Sofia University "St. Kl. Ohridsky"
5 J.Bourchier, 1164 Sofia, Bulgaria

Email: dtolev@fmi.uni-sofia.bg